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Topological groundwater hydrodynamics

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Abstract

Topological groundwater hydrodynamics is an emerging subdiscipline in the mechanics of fluids in porous media whose objective is to investigate the invariant geometric properties of subsurface flow and transport processes. In this paper, the topological characteristics of groundwater flows governed by the Darcy law are studied. It is demonstrated that: (i) the topological constraint of zero helicity density during flow is equivalent to the Darcy law; (ii) both steady and unsteady groundwater flows through aquifers whose hydraulic conductivity is an arbitrary scalar function of position and time are confined to surfaces on which the streamlines of the flow are geodesic curves; (iii) the surfaces to which the flows are confined either are flat or are tori; and (iv) chaotic streamlines are not possible for these flows, implying that they are inherently poorly mixing in advective solute transport.

1. Introduction

Theoretical subsurface hydrology is concerned with the predictive mathematical description of water flow and solute transport through heterogeneous porous media. Two principal lines of thinking have guided the development of this subdiscipline over the past two decades. One of them appropriates the language and conceptual underpinnings of fluid dynamics (itself a subdiscipline of continuum mechanics) to formulate partial differential equations containing parameters whose spatial and temporal dependence is prescribed pointwise in accordance with an observed or an imposed variability in the properties of a porous medium [1]. The other viewpoint takes its language and concepts from the theory of fluid turbulence to formulate stochastic partial differential equations whose parameters are modeled randomly in accordance with geostatistical inference from field-scale studies of a porous medium [2]. Both of these methodologies, deterministic and stochastic, have been applied with success to a variety of important problems in vadose-zone and groundwater hydrology, thanks in no small measure to the availability of high-performance computational algorithms and hardware [3–5].

It is important to bear in mind, however, that stochastic theories of subsurface flow and transport, while forgoing any details about the pointwise structure of spatial heterogeneity in porous formations, are not thereby rendered wholly random constructs, immune to the constraints and mechanisms implied by fluid dynamics, because this physical information still must be used to select the correct probabilistic ensemble with which to calculate observable quantities [6]. For example, any stochastic theory of tracer plume movement through an aquifer must utilize model probability density functions that respect solute mass conservation and incorporate properly the known physics of passive solute advection in a heterogeneous porous medium [7–9]. Otherwise, there is a decided risk of spurious parametric modeling of the plume spatial moments, which in fact have no intrinsic stochastic character, but are entirely deterministic quantities whose time evolution is governed by Darcy-scale fluid dynamics [10,11].

Recent theoretical research on fluid motions has brought into sharper view the *geometric* constraints that fluid kinematics alone impose on complex flow behavior [12–14]. Knowledge of these constraints is important in the context of fluid mixing, a purely advective process in which material filaments and surfaces in a fluid are stretched and folded, leading to complexity in their spatial configuration and to the spreading of a passively advected solute plume [11,12], an essential precursor to solute dilution [15,16]. The generality of the geometric

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approach to fluid motions derives from its use of purely kinematical concepts that are independent of constitutive properties (such as viscosity) or dynamical variables (such as pressure), [17] and from its focus on the *topology* of fluid flows, i.e., geometrical characteristics of fluid pathlines in three-dimensional space that persist despite the continual deformation of fluid volume elements during motion [18]. As pointed out by Aref [13], the passive advection of a solute by fluid motions is described mathematically by an ordinary differential equation relating fluid pathlines to the fluid velocity field – a kinematical relationship – and the question as to whether these pathlines can become chaotic – which greatly enhances solute mixing [11–13] – is predicated on the negative outcome of a search for topological constraints that would preclude chaos.

The present paper is an investigation of geometric constraints of this kind for groundwater flows governed by the Darcy law, under the condition that hydraulic conductivity and hydraulic head are continuously differentiable scalar functions of position and time coordinates. In Section 2, an important topological invariant, *helicity* [19], is found to have a one-to-one mathematical relationship to the Darcy law for both steady and unsteady groundwater flows. In Section 3, this relationship is shown to lead to the conclusion that groundwater flows are constrained always to lie on surfaces that either are flat or are surfaces of revolution, irrespective of the details of aquifer spatial variability as expressed in the dependence of the hydraulic conductivity on position coordinates. This broad result extends a similar conclusion for steady groundwater flows based on an analysis of the gradient of hydraulic conductivity in a heterogeneous aquifer [20]. In the present paper, the method of proof is less abstract and more inclusive (it encompasses unsteady groundwater flows), relying on some basic concepts in differential geometry that are outlined in Appendix A. In Section 4, a condition for the existence of chaotic pathlines in a smooth groundwater velocity field is defined and then shown to be precluded by the existence of the surfaces described in Section 3. Thus, groundwater flows governed by the Darcy law are geometrically constrained against becoming chaotic, thereby severely restricting their ability to spread a solute plume by mixing. Stochastic theories must necessarily reflect this constraint in the construction of model probability density functions that represent groundwater flow and transport.

2. Helicity

Helicity is defined [19] as the integral of the scalar product of vorticity with velocity over a region of space enclosed by a material surface (i.e., a surface that moves with a fluid as it flows and, therefore, always contains within it the same set of moving spatial points advected by the fluid velocity field [21]). The meaning of helicity can be understood physically by consideration of the prototypical steady (Eulerian) velocity field [22]:

$$\vec{v}(\vec{x}) = \vec{U} + \frac{1}{2}(\vec{\Omega} \times \vec{x}), \tag{1}$$

where $\vec{U}$ and $\vec{\Omega}$ are uniform vectors and $\vec{x}$ denotes a spatial point in a fluid. The fluid acceleration corresponding to this velocity field is found by calculating the material derivative of both sides of Eq. (1):

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} = \frac{1}{2}(\vec{\Omega} \times \vec{v}). \tag{2}$$

Eq. (2) implies that the velocity vector simply rotates with the angular speed $\frac{1}{2}\Omega$ around an axis along the fixed direction of $\vec{\Omega}$. Moving spatial points in the fluid thus follow paths resulting from the superposition of this rotary motion onto the constant rectilinear motion produced by the fixed velocity $\vec{U}$. Consistent with this picture are the solenoidal character of $\vec{v}(\vec{x})$ and of its vorticity $\vec{\omega} \equiv \vec{\nabla} \times \vec{v} = \vec{\Omega}$, both properties following directly from Eq. (1). The helicity density [22] in this example is accordingly a uniform scalar quantity

$$\vec{\omega} \cdot \vec{v} = \vec{\Omega} \cdot \vec{U}. \tag{3}$$

The geometric implications of Eq. (3) become transparent after applying conventional equations of differential geometry [23] to calculate the *curvature* K and *torsion* T of the streamlines of $\vec{v}(\vec{x})$:

$$K \equiv \frac{\|\vec{v} \times \mathrm{D}\vec{v}/\mathrm{D}t\|}{v^3} = \frac{1}{2}\left[\left(\frac{\Omega}{v}\right)^2 - \left(\frac{\vec{\Omega}\cdot\vec{U}}{v^2}\right)^2\right]^{1/2}, \tag{4a}$$

$$T \equiv \frac{(\vec{v} \times \mathrm{D}v/\mathrm{D}t)}{\|\vec{v} \times \mathrm{D}\vec{v}/\mathrm{D}t\|^2} \cdot \frac{\mathrm{D}^2\vec{v}}{\mathrm{D}t^2} = \frac{1}{2}\,\frac{\vec{\Omega}\cdot\vec{U}}{v^2}. \tag{4b}$$

If $K = 0$, the fluid velocity and acceleration are parallel vectors and the streamlines are merely straight lines [23]. This would occur if $\Omega = 0$ in Eq. (4a). If $T = 0$ (and $K \neq 0$), $\vec{v}$, $\mathrm{D}\vec{v}/\mathrm{D}t$, and $\mathrm{D}^2\vec{v}/\mathrm{D}t^2$ are coplanar vectors [23] and the streamlines must lie in a plane, which occurs in the present example if the helicity density is everywhere equal to zero. Thus, helicity endows streamlines with torsion, twisting them around an axis along the direction of fluid advection, and winding them into helical curves in three-dimensional space [22]. The bending of these curves, as measured by K in Eq. (4a), depends on the component of their vorticity $\vec{\Omega}$ that is not along the direction of advection by the uniform velocity $\vec{U}$. We note that both K and T in Eqs. (4a), (4b) are invariants of the fluid motion described by Eq. (1) because v^2 is an invariant of the fluid motion [$\mathrm{D}v/\mathrm{D}t = 0$ follows from Eq. (2)]. But the only possible twisted curve with constant K and T is the circular helix [23]. Some concepts of differential geometry that are useful to the interpretation of streamline topology are summarized in Appendix A.

The most important topological characteristic of groundwater flows governed by the Darcy flux law, [1]

$$\vec{J}(\vec{x},t) = -K(\vec{x},t)\vec{\nabla}\phi(\vec{x},t), \tag{5}$$

(where $\vec{J}$ [LT^{-1}] is volumetric flux density, K [LT^{-1}] is hydraulic conductivity, and ϕ [L] is hydraulic head) is the *complete absence of helicity*. This property emerges directly from a calculation of the velocity and vorticity fields corresponding to Eq. (5):

$$\vec{v}(\vec{x},t) \equiv \vec{J}(\vec{x},t)/\theta(\vec{x},t) = -(K/\theta)\vec{\nabla}\phi, \tag{6}$$

$$\vec{\omega}(\vec{x},t) \equiv \vec{\nabla}\times\vec{v} = \vec{\nabla}\phi\times\vec{\nabla}(K/\theta), \tag{7}$$

where $\theta(\vec{x},t)[L^3L^{-3}]$ is the porosity of the medium through which groundwater moves. The helicity density in a groundwater flow is calculated as the scalar product,

$$\vec{\omega}\cdot\vec{v} = -(K/\theta)\vec{\nabla}\phi\cdot(\nabla\phi\times\vec{\nabla}(K/\theta)) \equiv 0, \tag{8}$$

where use is made of a standard identity in vector analysis [20,23], $\vec{A}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{A}\times\vec{B}) \equiv 0$. Note that vorticity exists in a groundwater flow because of spatial inhomogeneity in the physical properties of the porous medium Eq. (7) [1].

Helicity is a kinematical property of broad physical significance in fluid mechanics, because it provides a quantitative measure of spatial complexity in flow fields and it remains constant as material elements (those always comprising the same set of spatial points advected by a velocity field [21]) are continuously deformed through fluid motions [22]. In groundwater flow fields governed by the Darcy law in Eq. (5), this constancy of helicity is perforce true, because the helicity density is identically equal to zero everywhere, quite irrespective of whether the flow is steady.

Eq. (8) indicates an absence of local helical motions in a groundwater flow. The effect of vorticity $\vec{\omega}$ on a flow field is to induce local rotation of a streamline around an axis specified by the direction of $\vec{\omega}$ [21]. If the vorticity is everywhere perpendicular to the fluid velocity, as in Eq. (8), there can be no rotation of the latter vector around an axis along the direction of flow and, therefore, no winding of streamlines into helical strands [19]. All local rotations of the velocity vector are confined to a plane perpendicular to its vorticity, making complex, tangled flow fields impossible [19]. Eq. (8) also is the condition both necessary and sufficient for any velocity field $\vec{v}(\vec{x},t)$ to be proportional to the gradient of a scalar function [1], a fact evident from combining Eqs. (7) and (8), noting that $\vec{\omega}(\vec{x},t)$ is always a solenoidal vector field [21]. Therefore, *the topological constraint, that a groundwater velocity field exhibits zero helicity density everywhere, is equivalent to the Darcy law*.

3. Lamb surfaces

3.1. Existence

Lamb surfaces are smooth, orientable two-dimensional manifolds that contain both the streamlines and the vorticity lines in a spatial domain of flow, excluding sources, sinks, and stagnation points [17]. These surfaces were discovered by Lamb [24], who proved their existence in any steady flow of an inviscid, incompressible fluid (see also the more technical proofs given by Poincaré [25] and Arnol'd [26]). Koslov [27] generalized this result to include homentropic fluids (those with uniform entropy fields) and Sposito [28] has done the same for any steady fluid flow in which vorticity field lines are material (i.e., field lines that always contain the same set of moving spatial points advected by the velocity field [21]). In these cases, Lamb surfaces partition the spatial domain of flow into nonintersecting, material streamsurfaces, the normal vectors to which are parallel with $\vec{\omega}\times\vec{v}$, accordingly termed the *Lamb vector* [17].

A continuously differentiable vector field is everywhere parallel to a family of surface-normals if and only if it is everywhere perpendicular to its curl [23]. Thus, for example, Eq. (8) is necessary and sufficient for the Darcy velocity vector $\vec{v}(\vec{x},t)$ to be everywhere parallel with the normal vectors to the equipotential surfaces (level surfaces) of the hydraulic head, as is well known [1]. The existence of Lamb surfaces imposes the same mathematical condition on the Lamb vector [17]

$$\vec{\nabla}\times(\vec{\omega}\times\vec{v})\cdot(\vec{\omega}\times\vec{v}) = 0. \tag{9}$$

Sposito [28] has suggested that Eq. (8) implies Eq. (9), based on the fact that a network of smooth orthogonal curves is always sufficient to define a surface on which they are its lines of curvature. *Lines of curvature*, by definition [29], follow along the directions in which the bending of a surface is extremal (e.g., the meridians and the parallels on a surface of revolution) and, therefore, they prescribe the shape of a surface in three-dimensional space. A direct demonstration of equivalence between Eqs. (8) and (9) will be given in the present paper, inspired by an argument due originally to Forsyth [30].

Let $\vec{q}$ be a unit vector tangent at time t to a streamline at a point $\vec{x}$ in the spatial domain of a groundwater flow [i.e., $\vec{v}(\vec{x},t) \equiv v(\vec{x},t)\vec{q}(\vec{x},t)$] and similarly let $\vec{m}$ be a unit vector tangent at time t to a vorticity line that intersects the streamline orthogonally at the same point $\vec{x}$ [i.e., $\vec{\omega}(\vec{x},t) = \vec{\nabla}\times\vec{v} \equiv \omega(\vec{x},t)\vec{m}(\vec{x},t)$]. Now let $\vec{a}(\vec{x},t)$ be a unit vector lying in the plane that is perpendicular to $\vec{m}(\vec{x},t)$ and let $\vec{b}(\vec{x},t)$ be a similar unit vector in the plane that is perpendicular to $\vec{q}(\vec{x},t)$, as illustrated in Fig. 1. Finally, let $\vec{n} \equiv \vec{m}\times\vec{q}$ be a unit vector perpendicular at time t to both $\vec{m}$ and $\vec{q}$ at the point $\vec{x}$.

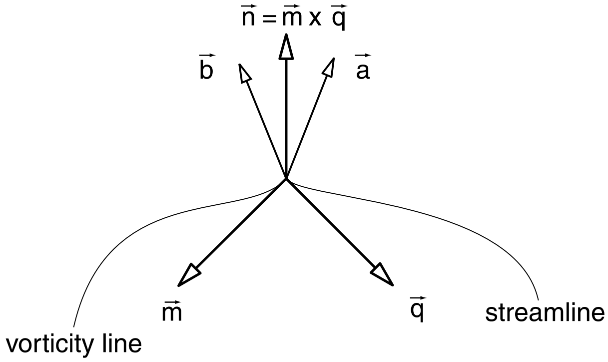

Fig. 1. An intersecting streamline and vorticity line, showing the unit tangent vectors $\vec{q}$ and $\vec{m}$, the unit normal vector $\vec{n}$ and the constructed vectors orthogonal to $\vec{q}$ and $\vec{m}$, $\vec{b}$ and $\vec{a}$, respectively.

This construction easily leads to the orthogonality conditions $\vec{q}\cdot\vec{b}=\vec{m}\cdot\vec{a}=\vec{q}\cdot\vec{n}=\vec{m}\cdot\vec{n}=0$. It is also straightforward then to demonstrate parallel directions between the vectors $\vec{q}$ and $\vec{n}\times\vec{b}$, as well as between $\vec{m}$ and $\vec{a}\times\vec{n}$:

$$\vec{q}\times(\vec{n}\times\vec{b})=(\vec{b}\cdot\vec{q})\vec{n}-(\vec{n}\cdot\vec{q})\vec{b}\equiv\vec{0}, \tag{10a}$$

$$\vec{m}\times(\vec{a}\times\vec{n})=(\vec{n}\cdot\vec{m})\vec{a}-(\vec{a}\cdot\vec{m})\vec{n}\equiv\vec{0}, \tag{10b}$$

where a standard identity for the triple vector product [23] has been applied in each case. The vectors $\vec{a},\vec{b}$, and $\vec{n}$ cannot be coplanar because $\vec{m}\cdot\vec{q}=0$. Therefore [23],

$$V\equiv(\vec{b}\times\vec{a})\cdot\vec{n}=\vec{a}\cdot(\vec{n}\times\vec{b})=\vec{b}\cdot(\vec{a}\times\vec{n}) \tag{11}$$

does not vanish at any time for any $\vec{a},\vec{b},\vec{n}$.

The objective now is to show that the vector $\vec{n}$ satisfies an "integrability condition", like Eq. (8), since, by hypothesis, $\vec{\omega}\times\vec{v}=\omega v\ \vec{n}$ and, therefore, Eq. (9) is the same as the expression

$$\vec{\nabla}\times\vec{n}\cdot\vec{n}=0. \tag{12}$$

Suppose that $\vec{n}$ shifts direction by an arbitrary infinitesimal amount at some instant. This shift, $\mathrm{d}\vec{n}$, can be expressed in terms of its components along the three mutually orthogonal unit vectors, $\vec{q},\vec{m}$, and $\vec{n}$:

$$\mathrm{d}\vec{n}=A\vec{q}+B\vec{m}+C\vec{n}, \tag{13}$$

where A,B, and C are scalar functions of $\vec{x}$ and t. Because $\vec{n}$ is a unit vector, $\mathrm{d}(\vec{n}\cdot\vec{n})=2\mathrm{d}\vec{n}\cdot\vec{n}=0$ and the coefficient $C\equiv0$. The remaining two coefficients can be determined by the equations:

$$\mathrm{d}\vec{n}\cdot\vec{a}=A\vec{q}\cdot\vec{a},\quad \mathrm{d}\vec{n}\cdot\vec{b}=B\vec{m}\cdot\vec{b}. \tag{14}$$

It is evident that only the component of $\mathrm{d}\vec{n}$ along the direction of $\vec{q}$ contributes to $\mathrm{d}\vec{n}\cdot\vec{a}$ and only that along the direction of $\vec{m}$ contributes to $\mathrm{d}\vec{n}\cdot\vec{b}$, given the orthogonality conditions on these vectors. Therefore, $\mathrm{d}\vec{n}$ in Eq. (14) can be replaced by two equivalent relations, respectively,

$$\mathrm{d}\vec{n}_q\equiv-k_q\vec{q}\,\mathrm{d}l,\quad \mathrm{d}\vec{n}_m\equiv-k_m\vec{m}\,\mathrm{d}\lambda, \tag{15}$$

$\mathrm{d}l$ and $\mathrm{d}\lambda$ being increments of arc length measured along a streamline and a vorticity line, respectively. The scalar coefficients, $k_q(\vec{x},t)$ and $k_m(\vec{x},t)$ $[\mathrm{L}^{-1}]$, are defined by Eq. (15). In general, both of Eqs. (15) would comprise two differential terms, one for each increment of arc length, reflecting the possibility that $\mathrm{d}\vec{n}_q/\mathrm{d}\lambda$ and $\mathrm{d}\vec{n}_m/\mathrm{d}l$ need not be equal to $\vec{0}$ [23]. These latter two derivatives, however, *are* equal to $\vec{0}$ if the streamlines and vorticity lines are both lines of curvature on a Lamb surface [23,29]. Eq. (15) anticipate this fact by defining only the scalar coefficients k_q and k_m, subject to their later interpretation after the proof of Eq. (12) is completed. In general, the product of k_q or k_m with a corresponding arc length increment, $\mathrm{d}l$ or $\mathrm{d}\lambda$, expresses the angle through which $\vec{n}$ is swept as it shifts infinitesimally along the direction of $\vec{q}$ or $\vec{m}$ [23].

The combination of Eqs. (10a), (10b), (11), (13), and (14) yields the differential expression:

$$\mathrm{d}\vec{n}=\frac{1}{V}[\mathrm{d}\vec{n}_q\cdot\vec{a}(\vec{n}\times\vec{b})+\mathrm{d}\vec{n}_m\cdot\vec{b}(\vec{a}\times\vec{n})]. \tag{16}$$

Eq. (16) is consistent with the constraints on $\mathrm{d}\vec{n}$; with the orthogonality conditions on $\vec{a},\vec{b}$, and $\vec{n}$; and with the expressions derived for the coefficients A and B. Calculation of the curl of $\vec{n}$ is then facilitated by using a dyadic equation [23] that is equivalent to Eq. (16):

$$\vec{\nabla}\vec{n}=-\frac{1}{V}[\vec{a}'k_q(\vec{n}\times\vec{b}')+\vec{b}'k_m(\vec{a}'\times\vec{n})], \tag{17}$$

where the primes,

$$\vec{a}'\equiv\vec{a}-(\vec{a}\cdot\vec{n})\vec{n},\quad \vec{b}'\equiv\vec{b}-(\vec{b}\cdot\vec{n})\vec{n} \tag{18}$$

define vectors that are still perpendicular to $\vec{m}$ or $\vec{q}$, respectively, but without components along $\vec{n}$. It is verified readily that V is invariant in value under substitution of $\vec{a}'$ and $\vec{b}'$ for $\vec{a}$ and $\vec{b}$, as are Eqs. (10a) and (10b). The use of the primed vectors is required in the dyadic $\vec{\nabla}\vec{n}$ in order to suppress terms arising from the operation, $\vec{n}\cdot\vec{\nabla}\vec{n}$, which has no counterpart in Eq. (16), but is a possible operation on the dyadic $\vec{\nabla}\vec{n}$. The definitions

$$\vec{q}\cdot\vec{\nabla}\vec{n}\equiv\frac{\mathrm{d}\vec{n}_q}{\mathrm{d}l}\quad \vec{m}\cdot\vec{\nabla}\vec{n}\equiv\frac{\mathrm{d}\vec{n}_m}{\mathrm{d}\lambda} \tag{19}$$

connect Eq. (17) to Eq. (16), given Eq. (15) and the orthogonality relations among $\vec{q},\vec{m}$, and $\vec{n}$.

The curl of $\vec{n}$ then follows conventionally [23] as the vector product of the pairs in the dyadic:

$$\begin{aligned}\vec{\nabla}\times\vec{n}&=-\frac{1}{V}[k_q\vec{a}'\times(\vec{n}\times\vec{b}')+k_m\vec{b}'\times(\vec{a}'\times\vec{n})]\\&=\frac{1}{V}[(k_m-k_q)(\vec{a}'\cdot\vec{b}')\vec{n}+k_q(\vec{n}\cdot\vec{a}')\vec{b}'\\&\quad-k_m(\vec{n}\cdot\vec{b}')\vec{a}'].\end{aligned} \tag{20}$$

Therefore,

$$\vec{\nabla} \times \vec{n} \cdot \vec{n} = \frac{k_m - k_q}{V} [(\vec{a}' \cdot \vec{b}') - (\vec{n} \cdot \vec{a}')(\vec{n} \cdot \vec{b}')]. \tag{21}$$

The significance of this result is evident after enlisting Eqs. (10a),(10b) to calculate the scalar product of $\vec{m}$ and $\vec{q}$

$$\begin{aligned} \vec{m} \cdot \vec{q} = 0 &= (\vec{a} \times \vec{n}) \cdot (\vec{n} \times \vec{b}) = (\vec{a}' \times \vec{n}) \cdot (\vec{n} \times \vec{b}'), \\ &= (\vec{a}' \cdot \vec{n})(\vec{n} \cdot \vec{b}') - (\vec{a}' \cdot \vec{b}'), \end{aligned} \tag{22}$$

where the last step again makes use of a standard identity in vector algebra [20,23]. Eq. (22) implies that the right side of Eq. (21) is zero and, therefore, that Eq. (9) is demonstrated, given that $\vec{\omega} \times \vec{v} = \omega v \, \vec{n}$. Note the key role of perpendicularity between $\vec{\omega}$ and $\vec{v}$ in proving that the Lamb vector is parallel to a surface-normal.

3.2. Geometry

The coefficients k_q, k_m in Eq. (15) are termed *principal normal curvatures* of the streamline and vorticity line to which they refer [23,29]. Formally, these parameters each are equal to the product of the curvature [Eqs. (4a) and (A.1)] of a vector line with the cosine of the angle between the *principal normal vector* of the vector line [$\vec{N}$ in Eq. (A.1)] and the unit normal vector to a surface that contains the vector line ($\vec{n}$), this angle being measured conventionally along a counterclockwise direction around an axis through the unit tangent vector to the line (i.e., $\vec{q}$ or $\vec{m}$). Thus, if the thumb of the right hand points along the unit tangent vector, the fingers of the hand curl in the direction along which the angle is measured from $\vec{N}$ to $\vec{n}$ [23]. In the present case the surface to which $\vec{n}$ is the normal vector is a Lamb surface.

The *Gaussian curvature* of a Lamb surface is equal to the product, $k_m k_q$ [23,31]. This topological characteristic retains its value at a given point on a surface irrespective of any smooth deformation of the surface that does not involve stretching, shrinking, or tearing [23,29,31]. Under this kind of continuous deformation (termed *bending*), and evidently under any wholesale translation or rotation of a surface, the distance between two points on a line on the surface and the angle between any two unit vectors on it remain unaffected along with the Gaussian curvature [23,29,31].

Points on a surface are termed *elliptic* if they have positive Gaussian curvature and *hyperbolic* if they have negative Gaussian curvature [29]. Spheres are closed surfaces with constant positive Gaussian curvature (the only ones with this intrinsic geometric property), equal to the inverse-square of their radii [29]. Surfaces of revolution can have either a positive or negative Gaussian curvature, which varies in value only along a meridian [29]. A surface is termed *flat* if its Gaussian curvature is zero [29,31]. If the flatness derives from only one of the principal curvatures having zero value, the points on the surface are *parabolic* (an example being the cylinder), whereas if both principal curvatures are zero, the points on the surface are *planar* [29]. Flat surfaces are those which can be bent into a plane through some continuous deformation process. They include cylinders, cones, planes, and the surfaces generated by a sequence of tangent lines to a curve in space [23,29]. It is a direct consequence of Eq. (15) that the surfaces swept out by moving the Lamb vector along either a vorticity line or a streamline *are always flat* [29].

Sposito [28] and Marris [32] independently have demonstrated an important geometric property of Lamb surfaces bearing steady fluid flows with zero helicity density everywhere: the streamlines on them are *geodesics*. This means that the distance between any two sufficiently close points on a Lamb surface is shortest when measured along a streamline connecting them as compared to any other curve that can be drawn between the two points [29,31]. Physically, this minimal characteristic exists because the groundwater speed [the magnitude of the vector $\vec{v}$ in Eq. (6)] is uniform along vorticity lines [28], implying that moving points on streamlines emanating from a single vorticity line reach any other single vorticity line intersecting them at the same instant, an identifying property of geodesics and their parallels [29]. Given the standard kinematic identity [21]

$$\frac{\mathrm{D}\vec{v}}{\mathrm{D}t} \equiv \vec{\nabla} \frac{1}{2} v^2 + \vec{\omega} \times \vec{v} \tag{23}$$

for the acceleration in any steady fluid motion, it follows that the forces in a steady groundwater flow must always act perpendicularly to the vorticity vector: $\vec{\omega} \cdot \mathrm{D}\vec{v}/\mathrm{D}t = 0$. Therefore, moving points in such a flow are accelerated only along streamlines or along Lamb vectors.

A technical summary of the proof that streamlines are geodesics on Lamb surfaces is given in Appendix A. There it is shown that this property of the streamlines implies that the principal normal vector $\vec{N}$ is always parallel to the surface normal vector $\vec{n}$, which then makes the principal normal curvature k_q become identical to the curvature K_q of a streamline [Eqs. (4a) and (A.1)]. Moreover, this intrinsic property of geodesics also means that the streamlines have zero torsion [as defined in Eqs. (4a) and (A.3)] and, therefore, that *the streamlines of a steady groundwater flow always lie in planes*. (The absence of streamline torsion can be appreciated in light of the fact that the twisting of a line of curvature is what causes the angle between $\vec{N}$ and $\vec{n}$ to vary at different points on a Lamb surface [23].) This important constraint is, in fact, sufficient to show that *Lamb surfaces for steady groundwater flows are either surfaces of revolution or are flat*. This result, whose derivation also is outlined in Appendix A, is consonant with the topological theorem [26,33] that Lamb surfaces for smooth, steady fluid flows that are confined to a bounded region of three-dimensional space must be either tori or cylinders, both of which have lines of curvature that lie in planes [30].

A vorticity line on a Lamb surface bearing a zero helicity fluid flow will be a geodesic on that surface only if there is no stretching or shrinking of the vorticity line by the velocity field, [28]. This condition is met if the velocity field is confined to a family of parallel planes in space (or to a single plane, as in many groundwater flow problems exhibiting translational or rotational symmetry [1]). Although this behavior cannot be presumed to exist universally in steady groundwater flows, Sposito and Weeks [8] found it to obtain for steady flows through the Borden aquifer as a consequence of the strong stratification of hydraulic conductivity with depth. The Lamb surface in this case was a cylinder, with a streamline as its base curve and a vorticity line as its linear generator. Quite generally, *if both streamlines and vorticity lines are geodesics on a Lamb surface, it will always be flat*. A technical summary of the proofs for these conclusions about the geometry of Lamb surfaces also is given in Appendix A.

4. Chaotic streamlines

Eq. (9) implies the existence of a scalar function $H(\vec{x},t)$ whose gradient is proportional to the Lamb vector and whose values define the level surfaces $[H(\vec{x},t) = \text{constant}]$ in three-dimensional space upon which streamlines and vorticity lines both lie:

$$\vec{\nabla} H = \mu(\vec{\omega} \times \vec{v}), \tag{24}$$

where $\mu(\vec{x},t)$ is an integrating factor [23] for the Lamb vector, $\vec{\omega} \times \vec{v}$. Evidently $\vec{v} \cdot \vec{\nabla} H = \vec{\omega} \cdot \vec{\nabla} H = 0$, illustrating the uniformity of $H(\vec{x},t)$ on streamlines and vorticity lines. For the case of steady fluid flow, the inverse of integrating factor $\mu(\vec{x})$ is equal to the Lagrangian vorticity [$\omega(\vec{x},t)$ is an Eulerian vorticity that is related to its Lagrangian counterpart through a conventional Piola transformation [33,34]]. For any fluid flow exhibiting steady material vorticity lines with $\mathrm{D}\vec{v}/\mathrm{D}t$ parallel to a surface-normal [i.e., $\mathrm{D}\vec{v}/\mathrm{D}t$ satisfies an "integrability" condition like that in Eq. (9)], the inverse of $\mu(\vec{x},t)$ is also equal to the Lagrangian vorticity [35].

Eq. (23) imposes severe constraints on the spatial distribution of fluid streamlines. Although streamlines may meander in complex ways on a Lamb surface, they can never leave it and, therefore, they cannot wander over time through the entire volume of a three-dimensional flow domain to explore it exhaustively. By contrast, streamlines that are not confined to sets of smooth nonintersecting surfaces in three-dimensional space are termed *chaotic* [14,36]. Chaotic streamlines exhaustively fill a spatial domain of flow [12,13], allowing the solutes they advect to mix well throughout the three-dimensional region the flow explores.

Evidently streamlines cannot become chaotic if $H(\vec{x},t)$ is nonuniform within a domain of flow, for then $\vec{\nabla} H \neq \vec{0}$ and Eq. (23) applies. However, if $\vec{\nabla} H \equiv \vec{0}$ in the domain of flow, Lamb surfaces are vitiated. In this case, the flow is termed *Beltrami* [12,17] and Eq. (23) leads to the condition:

$$\vec{\omega}(\vec{x},t) = c(\vec{x},t)\vec{v}(\vec{x},t), \tag{25}$$

where $c(\vec{x},t)$ is a smooth scalar function subject to the constraint,

$$\vec{v} \cdot \vec{\nabla} c + c(\vec{x},t)\vec{\nabla} \cdot \vec{v} = 0 \tag{26}$$

as follows from the solenoidal nature of $\vec{\omega}(\vec{x},t)$.

If a fluid motion is steady, then Eq. (25) implies that the product cJ is uniform on streamlines and vorticity lines, where $J > 0$ is the determinant of the deformation gradient tensor that connects a material line element in a Lagrangian reference configuration to its counterpart in the present configuration of the fluid [17,34]. (Recall that $\vec{v} \cdot \vec{\nabla} J = J\vec{\nabla} \cdot \vec{v}$ in a steady flow [21].) In this case, streamlines will lie on level surfaces determined by constant values of cJ and, therefore, they cannot become chaotic. It follows that cJ must be uniform throughout the domain of flow ($\vec{\nabla} cJ \equiv \vec{0}$) if chaotic streamlines are to exist. The present result extends a theorem of Ginzburg and Khesin [14,36], who showed that Eq. (24) with a uniform c is a necessary condition for chaotic streamlines in steady fluid flows for which Eq. (23) applies with $\mu = 1$, $J = 1$ (e.g., inviscid, incompressible fluids). A comparison of Eqs. (8) and (24) then leads to the conclusion that *chaotic streamlines are not possible for groundwater flows governed by the Darcy law* [Eq. (5)]. Vorticity and velocity vectors cannot be simultaneously perpendicular and parallel; a flow with vorticity and zero helicity density everywhere can never be Beltrami.

5. Conclusion

Groundwater flows through heterogeneous aquifers whose hydraulic conductivity is a scalar function of position and time coordinates are characterized topologically by zero helicity density everywhere. This geometric constraint is equivalent to the Darcy law as expressed in Eq. (5). It is also a sufficient condition to prove that groundwater flows governed by Eq. (5) lie on smooth surfaces containing the streamlines and vorticity lines of the flows. The streamlines are geodesic curves on these surfaces and, if the flow is steady, they lie in planes. Thus, the surfaces either are flat (e.g., a cylinder, as found for the Borden aquifer by Sposito and Weeks [8]) or they are tori. This strong restriction of groundwater flows to a surface, imposed by the zero helicity density constraint, means also that they cannot have chaotic streamlines and, therefore, they are inherently poorly mixing for advective solute transport. We note in passing that this conclusion does not apply to zero helicity flows in general (i.e., those for which the space integral of $\vec{\omega} \cdot \vec{v}$ is zero while $\vec{\omega} \cdot \vec{v} \neq 0$), since chaotic zero helicity flows have been identified, [39].

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Appendix A. Differential geometry of streamlines and lamb surfaces in groundwater flows

The fundamental theorem of the local theory of curves in three-dimensional space [37] states that the curvature and torsion parameters of any smooth oriented curve are sufficient to characterize it uniquely, except for its absolute position and orientation. It is this theorem that can be applied to show that Eqs. (4a), (4b) and the invariance of the speed v under fluid motion lead to helical streamlines.

By definition, the direction of a unit tangent vector $\vec{q}$ to a streamline is that along which arc length increases with time [29]. Changes in this direction occur if the streamline is not rectilinear, and these changes are quantified by its *curvature* [29,31]

$$\frac{D\vec{q}}{Dt} \equiv vK_q\vec{N}, \tag{A.1}$$

where $\vec{N}$ is a unit vector, termed the *principal normal vector* to the streamline, and K_q is the curvature of the streamline. The left side of Eq. (A.1) prescribes the derivative of $\vec{q}$ with respect to time following the motion along the streamline to which $\vec{q}$ is tangent [29,37]. The right side of Eq. (A.1) prescribes a vector along the direction of

$$\left[\left(\vec{v} \times \frac{D\vec{v}}{Dt}\right) \times \vec{v}\right]$$

[29], which is the component of fluid acceleration that lies in a plane perpendicular to $\vec{v}$. If this component is equal to zero, the direction of $\vec{q}$ is constant and the fluid motion is rectilinear, the signature of which is $K_q = 0$ [37].

A second direction perpendicular to $\vec{q}$ is specified by that along the vector $\vec{v} \times \frac{D\vec{v}}{Dt}$ or, equivalently $\vec{q} \times \frac{D\vec{q}}{Dt}$. It is defined formally by the vector product [29,37].

$$\vec{B} \equiv \vec{q} \times \vec{N} \tag{A.2}$$

and is termed the *binormal vector* to the streamline. The change in direction of $\vec{B}$ with time is described analogously to Eq. (A.1) [29,31]:

$$\frac{D\vec{B}}{Dt} \equiv -vT_q\vec{N}, \tag{A.3}$$

where T_q is the *torsion* of the streamline. (That the direction of $D\vec{B}/Dt$ is along $\vec{N}$ follows after calculating the material derivative of $\vec{q} \cdot \vec{B}$, noting that it must equal zero and that $D\vec{B}/Dt$ must always be perpendicular to $\vec{B}$ because the latter is a unit vector.) If $\vec{B}$ does not change direction, by Eq. (A.2) the streamline must always lie in the plane containing $\vec{q}$ and $\vec{N}$. This plane is then fixed in space because of Eq. (A.1) and the restriction a constant $\vec{B}$ imposes on $D\vec{N}/Dt$ to be parallel with $\vec{q}$, as follows from calculating the material derivative of both sides of Eq. (A.2) [29]. The signature of a planar fluid motion is thus $T_q = 0$ in Eq. (A.3). Note that Eqs. (A.1) and (A.3) may be summarized in a generic expression [23,28]:

$$\frac{D\vec{e}}{Dt} = v(\vec{\delta}_q \times \vec{e}) \quad (\vec{e} = \vec{q}, \vec{N}, \vec{B}), \tag{A.4}$$

where $\vec{e}$ is any unit vector either tangent ($\vec{q}$) or perpendicular ($\vec{N}, \vec{B}$) to a curve in space and

$$\vec{\delta} \equiv T_q\vec{q} + K_q\vec{B} \tag{A.5}$$

is termed the *Darboux vector* for $\vec{q}, \vec{N}, \vec{B}$. [23] The form of Eq. (A.4) shows that the vector $\vec{e}$ rotates with angular speed $v\delta_q = v(K_q^2 + T_q^2)^{1/2}$ around an axis along the direction of the Darboux vector, [23] Marris and Passman [38] and, more briefly, Sposito [28] have summarized the relationship of Eqs. (A.4) and (A.5) to the existence of Lamb surfaces for steady fluid flows with zero helicity density everywhere (e.g., steady groundwater flows).

Marris [32] demonstrates the existence of Lamb surfaces and derives their geometric properties for steady fluid flows governed by Eq. (8), starting from the premise that vorticity lines are material (his Theorem 3.3 [32]). The mathematical statement of this premise is termed the *Helmholtz–Zoraski criterion* [17].

$$[\vec{\nabla} \times (\vec{\omega} \times \vec{v})] \times \vec{\omega} = \vec{0} \tag{A.6}$$

from which Eq. (9) and the existence of Lamb surfaces are an immediate consequence. Eqs. (8) and (A.1) imply the condition, [32]

$$\vec{\nabla} \times (\vec{\omega} \times \vec{v}) \cdot \vec{q} = 0 \tag{A.7}$$

from whose expression in terms of the components of $\vec{\omega}$ Marris [32] demonstrates that

$$\vec{\omega} = (K_q v - \vec{N} \cdot \vec{\nabla} v)\vec{B}, \tag{A.8}$$

where K_q is the curvature and $\vec{N}$ is the principal normal vector of the streamline for which $\vec{\omega}$ is the curl; i.e.,

$$\vec{q} \cdot \vec{\nabla}\vec{q} = K_q\vec{N} \tag{A.9}$$

and $\vec{B}$ is the binormal vector of the streamline, defined in Eq. (A.2). Eq. (A.9) is the special case of Eq. (A.1) that is applicable to steady fluid flows. It can be related to Eq. (A.4) by calculating the gradient of $\vec{q} \cdot \vec{q}$ using the standard vector identities [23]:

$$\vec{\nabla}(\vec{A} \cdot \vec{A}) \equiv 2[\vec{A} \cdot \vec{\nabla}\vec{A} + \vec{A} \times (\vec{\nabla} \times \vec{A})], \tag{A.10a}$$

$$\vec{\nabla} \times \vec{A} \equiv (\vec{\nabla} \times \vec{A} \cdot \vec{A})\vec{A} + \vec{A} \times [(\vec{\nabla} \times \vec{A}) \times \vec{A}], \tag{A.10b}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) \equiv (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \tag{A.10c}$$

Eqs. (A.6) and (A.8) are sufficient to prove that $\vec{N}$ satisfies a condition of the same form as Eq. (8) and, therefore, that $\vec{N}$ is parallel to the Lamb vector, $\vec{\omega} \times \vec{v}$. [32] This means that the streamlines are geodesics of the Lamb surfaces on which they lie [23,29].

Sposito [28] derives Eq. (A.7) from the premise that the *Frobenius integrability condition* [40].

$$\vec{q} \cdot \vec{\nabla}\vec{m} - \vec{m} \cdot \vec{\nabla}\vec{q} = \gamma_q\vec{q} + \gamma_m\vec{m} \tag{A.11}$$

applies to the vector $\vec{q}$ and $\vec{m}$, where γ_q, γ_m are termed the *geodesic curvatures* of the intersecting streamline or vorticity line to which $\vec{q}$ or $\vec{m}$ is a unit tangent vector. Eq. (A.13) is necessary and sufficient for the existence of a Lamb surface through Eq. (9), to which it is equivalent, [28] The geodesic curvature γ_q is defined as the product of the curvature K_q Eq. (A.1) and the sine of the angle between $\vec{N}$ and the Lamb vector, this angle being positive when measured counterclockwise relative to an observer toward whom $\vec{q}$ points, [23] The same concept applies to γ_m. A curve is a *geodesic* on a surface if its geodesic curvature is zero, [29] a sufficient condition for which is that $\vec{N}$ and $\vec{m} \times \vec{q}$ are parallel vectors (i.e., zero sine of the angle between them). Sposito [28] demonstrated that $\gamma_q = 0$ follows from Eq. (8) and the Frobenius condition, whereas Marris [32] showed that $\vec{N}$ in Eq. (A.9) and $\vec{m} \times \vec{q}$ are parallel vectors.

The fundamental theorem of the local theory of surfaces in three-dimensional space [37] states that the geometric properties of any smooth surface (except for its absolute position and orientation) are determined fully by a set of three coupled partial differential equations known collectively as the *Mainardi–Codazzi equations* and the *Gauss equation*. In the present case, these equations take the form [23,28]:

$$\vec{m} \cdot \vec{\nabla}k_q = 0, \quad \vec{q} \cdot \vec{\nabla}k_m = \gamma_m(k_m - k_q) \quad \text{(Mainardi–Codazzi)} \tag{A.12}$$

$$\vec{q} \cdot \vec{\nabla}\gamma_m = \gamma_m^2 + k_m k_q \quad \text{(Gauss)} \tag{A.13}$$

given that streamlines and vorticity lines are lines of curvature on Lamb surfaces, and that the streamlines are geodesics on these surfaces as well if the velocity field exhibits zero helicity density everywhere. Because of this latter property of the streamlines, $k_q = K_q$, the curvature (i.e., the cosine of the angle between the Lamb vector and the principal normal vector of a streamline is equal to 1.0), and the first of Eq. (A.14) then shows that the curvature of the streamlines crossing a given vorticity line is the same at all points of intersection. Eq. (A.1) implies that curvature is a measure of the strength of fluid acceleration along the Lamb vector and, therefore, the uniformity of K_q along vorticity lines expresses that same property for the force causing streamlines to bend in three-dimensional space. We note in passing that the geodesic property of streamlines also implies that the Darboux vector [Eq. (A.5)] is parallel to the vorticity vector [Eq. (A.8)], thus imparting to K_q v a familiar physical meaning as the angular speed of rotation of a streamline around a vorticity line, [23] This angular speed is uniform along the latter because the streamlines crossing it are geodesic curves on a Lamb surface. After making the substitutions,

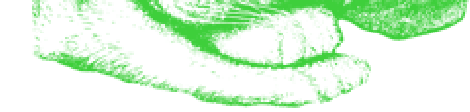
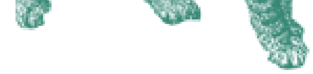
$$k_q = K_q, k_m = K_m \cos\theta_m, \gamma_m = K_m \sin\theta_m \tag{A.14}$$

in Eqs. (A.12) and (A.13), where θ_m is the angle between the principal normal vector of the vorticity line to which $\vec{m}$ refers and the Lamb vector, one can derive an ordinary differential equation for K_m, the curvature of a vorticity line:

$$\frac{dK_m}{d\theta_m}\frac{1}{K_m^2} = K_q \sin\theta_m. \tag{A.15}$$

This differential equation is satisfied by a surface of revolution obtained from rotating a planar curve (in the present case, a streamline) around the *z*-axis of a Cartesian coordinate system [23,29]. Eq. (A.15) is sufficient to conclude that the Lamb surface must be either a torus or a cylinder, because these are the only orientable smooth surfaces possible for a vector field that has no singularities or stagnation points, and which occupies a finite domain in three-dimensional space, [33]. The former surface is obtained by rotating a circle around the *z*-axis whereas the latter is the result of rotating a curve in the same fashion. Strictly speaking, the Lamb surface described by Eq. (A.15) need not look exactly like a torus or a cylinder, but need only be transformable into one of these two surfaces of revolution by a continuous deformation process.

Eq. (A.13), the Gauss equation for a surface [23], includes the Gaussian curvature, $k_m k_q$ [23,31]. If the vorticity lines on a Lamb surface are also geodesic curves, like the streamlines, their geodesic curvature $\gamma_m = 0$, by definition [29], and the two terms in Eq. (A.13) containing γ_m must then vanish identically [as will the right sides of Eqs. (A.11) and (A.15)]. It follows that $k_m k_q$ also must vanish in this case, implying a flat Lamb surface [23,29,31]. Eq. (A.15) then describes a cylinder. The Lamb surface accordingly will be the cylinder that comprises straight vorticity lines extending from a base curve that is formed by a streamline, as

observed in steady groundwater flows through the Borden aquifer [8].

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